Computational Complexity for Uniform Orientation Steiner Tree Problems

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Abstract

We present a straighforward proof that the uniform orientation Steiner tree problem, also known as the λ-geometry Steiner tree problem, is NP-hard whenever the number of orientations, λ, is a multiple of 3. We also briefly outline how this result can be generalised to every $\lambda > 2$.

Keywords: Steiner tree problem, λ-geometry, computational complexity, NP-hard.

1 Introduction

Given a set of points N and set of $\lambda \geq 2$ uniformly distributed (legal) orientations in the plane, we consider the problem of constructing a minimum-length tree that interconnects N with the restriction that the tree is composed of line segments using legal orientations only. The focus of this paper is mainly on the case where λ is a multiple of 3, however we will also discuss the problem for the more general $\lambda \geq 2$ case. This so-called *uniform orientation* (or λ-geometry) Steiner tree problem is equivalent to computing a minimum Steiner tree for N under a metric where the unit circle is a regular 2λ sided polygon. In the Steiner problem the interconnection network may contain nodes other than the points in N. Computing the optimal locations of these nodes and the topology of the network makes this a computationally challenging problem.

The uniform orientation Steiner tree problem has important applications in micro-chip design, where millions of nets need to be routed on a (small) number of chip layers. On each routing layer, all wires generally use the same orientation in order to make joint routing of multiple nets feasible. In optimising the routing, the design of each net is usually treated as a planar geometric optimisation problem in λ-geometry, where the cost of transition between layers is treated as negligable. Today, most chip design technologies use only two perpendicular routing orientations (the so-called Manhattan routing where $\lambda = 2$), but the increasing number of available routing layers has made the use of multiple orientations relevant in practice (Chen et al. 2005, Teig 2002).

One of the great challenges in the design of integrated circuits for micro-chips is the continuing increase in density of these circuits, where the number of transistors on a chip tends to double approximately every two years (an observation known as Moore's law). This means that it is essential not only to devise optimisation algorithms that allow the nets to be designed as compactly as possible, but also to understand the computational complexity of such algorithms, since the scaling of these problems is a major issue. It is this question of computational complexity that this paper addresses.

It is well-known that the λ-geometry Steiner tree problem is NP-hard for the rectilinear metric ($\lambda = 2$) (Garey & Johnson 1977) and the Euclidean metric ($\lambda \to \infty$) (Garey, Graham & Johnson 1977). More recently, an NP-hardness proof was given for the λ-geometry Steiner tree problem for $\lambda = 4$ (Müller-Hannemann et al. 2007); this proof adapts the proof for the Euclidean case to the $\lambda = 4$ case.

Rubinstein et al. (1997) have given an elegant proof of the NP-hardness of a special case of the Euclidean Steiner tree problem — where the terminals are restricted to lying on two parallel lines. This approach was adapted by Brazil et al. (1998) to show that the gradient constrained Steiner tree problem is NP-hard, and the arguments have been simplified and further generalised in a later paper (Brazil et al. 2000).

Our Contribution.

We show that a method similar to that pioneered by Rubinstein et al. (1997) can be applied to the λ-geometry Steiner tree problem, to show that the λ-geometry Steiner tree problem is NP-hard whenever λ is a multiple of 3. We also briefly outline the generalisation of this result to every $\lambda > 2$. All of these NP-hardness results are new, apart from the result for $\lambda = 4$ (the octilinear norm), and even in that case the proof is significantly simpler than the very technical proof given by Müller-Hannemann et al. (2007).

Organisation of the Paper.

In Section 2 we summarise a number of structural results for Euclidean and λ-geometry Steiner trees that are relevant for the NP-hardness proof. The main result follows in Section 3, with a focus on the case where λ is a multiple of 3. We conclude with a brief discussion of generalisations of these results.

2 Preliminaries

The well-known Euclidean Steiner tree problem is defined (as a decision problem) as follows:

EUCLIDEAN STEINER TREE PROBLEM
Instance: A finite set of points N lying in the Euclidean plane and a positive integer L.
Question: Is there a tree T interconnecting the set N such that the length of T is at most L?

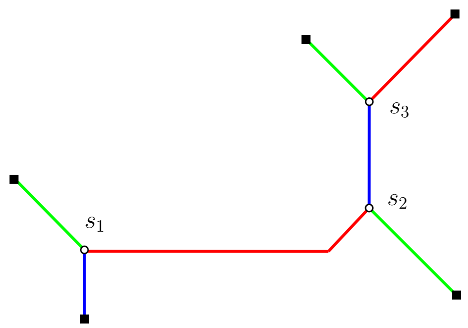

Figure 1: Full Steiner tree for five terminals and $\lambda = 4$. Black vertices are terminals and white vertices (s_1, s_2 and s_2) are Steiner points. The tree has six straight edges and one bent edge (s_1, s_2). Edges are colored by their colors in the corresponding direction set.

By the *length* of a tree T, we mean the sum of the lengths of the edges of T. A tree T with length L satisfying the Euclidean Steiner tree problem, where L is as small as possible for a given set N, is called a minimum Steiner tree for N. The given points N are called the *terminals* in T, and other vertices of T (of degree at least 3) are called *Steiner points*.

Let $\lambda \geq 2$ be a given positive integer. Given λ orientations $j\omega$ ($j = 1, 2, ..., \lambda$) in the Euclidean plane, where $\omega = \pi/\lambda$ is a unit angle, we represent these orientations by the angles with the x-axis of corresponding straight lines. A line or line segment with one of these orientations is said to be in a *legal* direction. Objects composed of line segments in legal directions are said to belong to a λ-geometry.

For any given λ-geometry we define the following Steiner tree problem:

λ-GEOMETRY STEINER TREE PROBLEM
Instance: A finite set of points N in the Euclidean plane and a positive integer L.
Question: Is there a λ-geometry Steiner tree T with terminal set N such that the length of T is at most L?

Again, a minimum length tree T satisfying this problem for a given set N is known as a λ-geometry minimum Steiner tree for N.

The main part of the NP-hardness proof makes use of some key properties of Euclidean Steiner tree problem (Gilbert & Pollak 1968) and λ-geometry Steiner tree problem (Brazil et al. 2006, 2009). These properties are summarised below.

2.1 Direction Sets in λ-Geometry

Consider a λ-Geometry minimum Steiner tree T for a given set N. The Steiner points in T necessarily each have degree 3 or 4. In our NP-hardness proof only Steiner points that have degree 3 are relevant; degree 4 Steiner points (which only exist in very restricted cases) cannot occur as part of the minimum Steiner trees for the instances we construct.

Edges in T consist of line segments that use either a single legal orientation (*straight* edge) or two neighbouring legal orientations (*bent* edge); in the latter case we may assume that the edge consists of exactly two line segments (separated by angle $\omega = \pi/\lambda$ and called half-edges) that meet at a corner point (Figure 1).

Consider the set of legal orientations of the line segments of the edges that are adjacent to some Steiner point s of degree 3 in T; more precisely, consider each line segment as being oriented away from s, and let D be the corresponding set of directions. The set D is denoted a *direction set* if it is maximal under inclusion, i.e., there exists no minimum Steiner tree with some Steiner point that has a set of directions that is a superset of D. Local optimality conditions at Steiner point imply that direction sets can be characterized precisely (Brazil et al. 2006, 2009) (see Figure 2); when λ is a multiple of 3, the direction set has 6 directions and otherwise it has has 4 directions. The first pair of directions forms the so-called red edge, and the other directions are part of the remaining green and blue edges. The red, green and blue edges are separated by angles that are as close to 120° as possible (Figure 1). The total number of possible direction sets is 2λ — one for each pair of possible assignment of neighbouring red directions.

λ	Directions	
$3m$	Red:	$0,\ \omega$
	Green:	$2m\omega,\ (2m+1)\omega$
	Blue:	$4m\omega,\ (4m+1)\omega$
$3m+1$	Red:	$0,\ \omega$
	Green:	$(2m+1)\omega$
	Blue:	$(4m+2)\omega$
$3m+2$	Red:	$0,\ \omega$
	Green:	$(2m+2)\omega$
	Blue:	$(4m+3)\omega$

Figure 2: Feasible directions in a direction set (up to rotation by a multiple of ω).

A minimum Steiner tree can be decomposed into components in which every terminal is a leaf, known as *full components*, or *full minimum Steiner trees*. This decomposition is unique for a given minimum Steiner tree, but is not unique for a given terminal set. A minimum Steiner tree is said to be *fulsome* if it has the maximum possible number of full components for the given terminal set. Hence, a minimum Steiner tree is full and fulsome if there is no minimum Steiner tree on the same set of terminals with two or more full components.

Our interest in direction sets stems from the fact that all Steiner points in a full minimum Steiner tree use the *same* direction set; more precisely, we have the following theorem (Brazil et al. 2006) — a generalisation of this theorem to general weighted fixed orientation metrics has been given by Brazil et al. (2009):

Theorem 2.1 *(Brazil et al. 2006) Given a fulsome full minimum Steiner tree in λ-geometry, there exists a* single *direction set that is used by* every *Steiner point in the tree (where direction sets that can be obtained from each other by reflecting all directions through the Steiner point are considered to be equivalent).*

2.2 Zero-Shifts in λ-Geometry

A consequence of Theorem 2.1 is that the edges in a full minimum Steiner tree can be colored red, green and blue in such a way that all edges with the same colour use the same orientations (either a single orientation or two neighbouring orientations). Let e be a straight edge or half-edge in a full minimum Steiner tree T, oriented in direction $j\omega$. Then e is said to be *primary* if $(j-1)\omega$ is not a feasible direction with the same colour as e. Similarly, e is said to be *secondary* if $(j+1)\omega$ is not a feasible direction with the same colour as e. If $\lambda \neq 3m$ then it is possible for an edge to be both primary and secondary. We say that e is *exclusively primary* (or *exclusively secondary*) if it is primary, but not secondary (or, respectively, secondary, but not primary).

A minimum Steiner tree T is usually not unique in λ-geometry since the metric is not strictly convex. We define

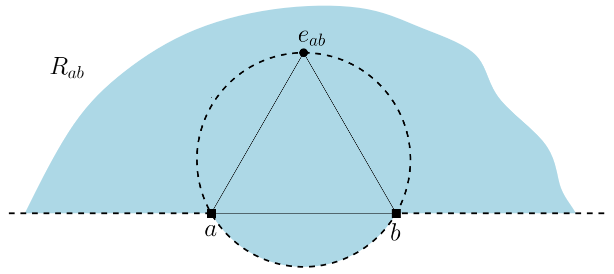

Figure 3: One of the two possible regions R_{ab} for two given points a and b. The other region is obtained by reflecting the diagram through the line through a and b.

a *zero-shift* as a perturbation of one or more Steiner points in T such that the perturbation does not increase the length of T. The identification of primary/secondary edges plays an important role for zero-shifts:

Theorem 2.2 *(Brazil et al. 2009) Let e_1 and e_2 be two edges in a full and fulsome minimum Steiner tree T such that e_1 has an exclusively secondary component and e_2 has an exclusively primary component. Then there exists a zero-shift acting on the Steiner points on the path from e_1 to e_2 in T, such that the shift can continue to be performed until either e_1 has no exclusively secondary component or e_2 has no exclusively primary component. Furthermore, this shift preserves the direction of all straight edges except (possibly) e_1 and e_2.*

A straightforward corollary of this theorem is that if, for a given set of terminals N, there exists a full and fulsome minimum Steiner tree, then there exists a minimum Steiner tree for N that has *at most one bent edge*.

2.3 Empty Regions for Euclidean Minimum Steiner Trees

Given two distinct points a and b in the Euclidean plane, let e_{ab} be the third vertex of an equilateral triangle with vertices a and b, let C_{ab} be the open finite region bounded by the circumcircle of $\triangle abe_{ab}$, and let R_{ab} be the union of C_{ab} and the open half plane defined by the line through a and b and containing e_{ab}, as illustrated in Figure 3. Note that R_{ab} is not uniquely defined; there are two possibilities for e_{ab} resulting in two possible choices for the region R_{ab}.

Proposition 2.3 *Let a and b be terminals of a Euclidean minimum Steiner tree T. If there exists a region R_{ab}, as defined above, containing no terminals of T then that region also contains no Steiner points of T.*

Proof. This is a simple extension of the "wedge property" introduced and proved by Gilbert & Pollak (1968). The wedge property states that any open wedge-shaped region having an angle of $2\pi/3$ and containing no terminals of T also contains no Steiner point of T. Region R_{ab} is an infinite union of such wedges, all with a and b on their boundary. ∎

3 NP-Hardness of the λ-Geometry Steiner Tree Problem

In this section we prove that the λ-geometry minimum Steiner tree problem is NP-complete for the cases where $\lambda = 3m$. We establish this result, in Corollary 3.2 below, by first proving a strictly stronger theorem, namely that the following class of problems is NP-complete for $\lambda = 3m$.

PARALLEL LINES λ-GEOMETRY STEINER TREE PROBLEM
Instance: A finite set of points N lying on two parallel lines in the Euclidean plane and a positive integer L.
Question: Is there a λ-geometry Steiner tree T with terminal set N such that the length of T is at most L?

In order to avoid issues related to the theoretical complexity of computing with exact real arithmetic, we in fact consider a *discretised* version of the problem as described later; in the construction below we initially ignore this technical difficulty.

We will show that for any given integer λ which is a multiple of 3 the PARALLEL LINES λ-GEOMETRY STEINER TREE PROBLEM is NP-complete. The main idea is to show that the problem can be used to polynomially encode an instance of the SUBSET SUM PROBLEM, which is well-known to be NP-complete (Garey & Johnson 1979):

SUBSET SUM PROBLEM
Instance: A set $S = \{d_1, \dots, d_n\}$ of integers and an integer d.
Question: Is there a set $J \subseteq \{1, \dots, n\}$ such that $\sum_{i \in J} d_i = d$?

The main result is as follows. In the proof we let $\mathbf{d}(a, b)$ represent the Euclidean distance between the points or parallel lines a and b.

Theorem 3.1 *The parallel lines λ-geometry Steiner tree problem is NP-complete for any given $\lambda = 3m$ (where m is a positive integer).*

Proof. Let $S = \{d_1, \dots, d_n\}$ and $d < \sum_{i=1}^{n} d_i := D$ be a given instance of the SUBSET SUM PROBLEM. We first show how to use this instance to construct (in polynomial time) an instance of the PARALLEL LINES λ-GEOMETRY STEINER TREE PROBLEM, and then show that the instance for the SUBSET SUM PROBLEM is a "yes" instance if and only if the corresponding instance for the PARALLEL LINES λ-GEOMETRY STEINER TREE PROBLEM is a "yes" instance. The statement of the theorem then follows.

The construction of the instance for the PARALLEL LINES λ-GEOMETRY STEINER TREE PROBLEM is as follows. We describe the construction in four stages:

1: Let V_1, V_1', V_2', V_2 be four vertical lines ordered from left to right such that

$$\mathbf{d}(V_1, V_2) \gg \mathbf{d}(V_1, V_1') = \mathbf{d}(V_2', V_2) \gg D. \quad (1)$$

Let u_0 be a fixed point on V_2, and construct a zigzag path P between u_0 and a point on V_1 (labelled v), such that: P is composed of line segments with alternating polar angles $2\pi/3$ and $\pi/3$; P has $2n$ internal vertices (where n is the cardinality of S); and these internal vertices lie alternatively on V_1' and V_2'. See Figure 4.

2: Now from each internal vertex of P on V_1' extend a horizontal line segment to a point on V_1. Label these n points x_1 to x_n in ascending order. Similarly, from each internal vertex of P on V_2' extend a horizontal line segment to a point on V_2, and label these points u_1 to u_n in ascending order. Again, this is illustrated in Figure 4. This results in a λ-geometry Steiner tree interconnecting u_0, the u_i's, x_i's and v (where in each case i runs from 1 to n). We call this tree the *base tree* T_x.

3: The next stage of the construction is to replace each point x_i by three points on V_1 labeled, from bottom to top, a_i, b_i and c_i, satisfying: $|a_ib_i| = d_i$; $|b_ic_i| = d_i$; and x_i is the midpoint of a_ib_i (Figure 5). We also alter the Steiner tree constructed in Stage 2, so that

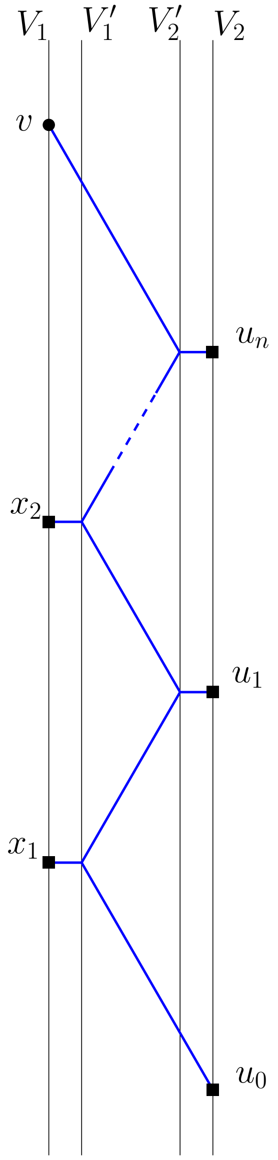

Figure 4: Construction for the case $\lambda = 3m$. The initial two stages of the construction result in the tree shown, the *base tree* T_x.

it connects to a_i, b_i and c_i, instead of x_i. This is done by shortening the horizontal edge by $d_i/(2\sqrt{3})$ on the left and creating a Steiner point at that new left endpoint with two new incident edges with polar angles $2\pi/3$ and $\pi/3$ and each with length $d_i/\sqrt{3}$ connecting to a_i and b_i. Finally we connect b_i to c_i with a single (geodesic) edge in λ-geometry, which is a vertical line segment (if m is even) or a bent edge using the two legal directions closest to vertical (if m is odd). This is illustrated in Figure 5(a), for the case where m is odd. Let N_v be the set consisting of u_0, the u_i's, a_i's, b_i's, c_i's and v. We denote the above λ-geometry Steiner tree (interconnecting the elements of N_v) by T_v. We will refer to the topology of the base tree T_x (from Stage 2) as the *base topology* of T_v.

Before completing the construction, we establish the following claim:

Claim 1. T_x and T_v are each the unique minimum Steiner λ-tree for their respective terminal sets.

Proof of Claim 1. Given the differences in scale in Inequality (1), consider the limiting case where $\mathbf{d}(V_1, V_1') = \mathbf{d}(V_2', V_2) = 0$. In that case each of T_x and T_v becomes a single zigzag path with polar angles $2\pi/3$ and $\pi/3$ between terminals $\{x_1, \ldots, x_n, v\}$ on V_1 and $\{u_0, u_1, \ldots u_n\}$ on V_2. The fact that this path is a Euclidean minimum Steiner tree (and hence a minimum Steiner λ-tree) on its vertices follows from Proposition 2.3 by constructing suitable regions: $R_{u_i u_{i+1}}$ for each $i \in \{0, \ldots, n-1\}$; $R_{x_i x_{i+1}}$ for each $i \in \{1, \ldots, n-1\}$; and $R_{x_n v}$. Taking the union of these regions, it is clear that any Steiner points must coincide with terminals, hence the minimum Steiner tree coincides with the minimum spanning tree. Furthermore, this minimum spanning tree is easily seen to be unique.

The result now follows immediately by continuity, and the fact that T_x and T_v (in the non-limiting case) are each locally minimal at every Steiner point. ∎

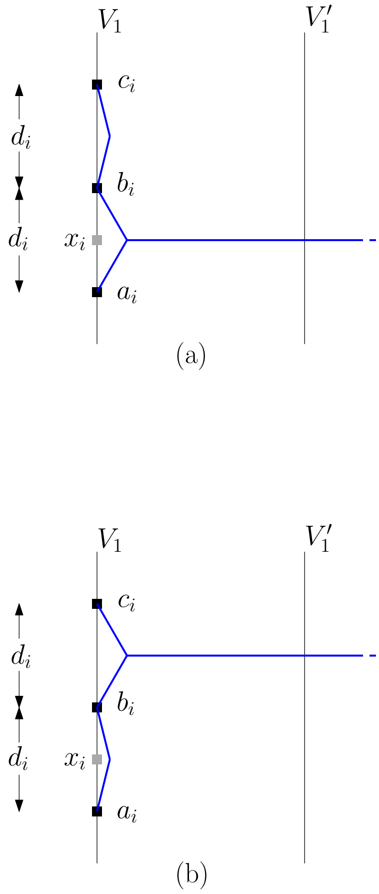

Figure 5: Stage 3 construction for the case $\lambda = 3m$. Diagram (a) shows how T_V connects to each triple a_i, b_i, c_i for the case where m is odd. Diagram (b) is the alternative connection possible in the tree T_0, used in Claim 2.

Note that it is straightforward to compute the total Euclidean length of T_v (i.e., $|T_v|$) in terms of $\mathbf{d}(V_1', V_2')$, $\mathbf{d}(V_1, V_1')$ and S. Let $L_v := |T_v|$. Also, we observe that the main full component of T_v, containing all the Steiner points, uses only three legal directions (and hence has no bent edges). We describe such a tree as a *3-direction Steiner tree*.

The final stage of our initial construction is as follows.

4: Let v_0 be the point on V_1 below v such that $|v_0 v| = 2d$. Let N_0 be the set N_v where v has been replaced by v_0. Let T_0 be a minimum Steiner λ-tree for N_0. In other words, we can think of T_0 as being the new minimum Steiner tree obtained from T_v by moving the terminal v vertically downwards by $2d$.

We next establish some properties of the tree T_0.

Claim 2. The minimum Steiner λ-tree T_0 has the same base topology as T_v. Furthermore, for each triple, a_i, b_i and c_i, the main full component of T_0 either connects directly to a_i and b_i only, as in Figure 5(a), or to b_i and c_i only, as in Figure 5(b).

Proof of Claim 2. The first statement follows by the relative scale of the distances involved in Inequality (1), using the same argument as in the proof of Claim 1. For the second statement, it is an easy exercise to show that the configurations shown in Figure 5(a) and (b) are the only locally minimal ways of connecting the main full component of T_0 to a_i, b_i and c_i. ∎

Claim 3. The following three statements are equivalent;

1. The answer to the given instance of the SUBSET SUM PROBLEM is "yes".
2. There exists a 3-direction minimum Steiner λ-tree on N_0 with the same base topology as T_v.
3. There exists a Steiner λ-tree on N_0 with length at most $L_v - \sqrt{3}d$.

Proof of Claim 3. The equivalence of the three statements is shown in four steps.

Step 1: (1) ⇒ (2). Let T_x be the minimum Steiner λ-tree constructed in Stage 2 of the main construction. Suppose we treat v and one of the terminals x_i as 'moveable' points, able to move along V_1. Then consider the following question: If we move x_i vertically upwards by a distance δ, how does the position of v on V_1 change so that T_x remains a 3-direction tree? As Figure 6 shows, each horizontal edge incident with a terminal u_j (for j such that $i \leq j \leq n$) increases in length by $2\delta/\sqrt{3}$. In particular, the horizontal edge incident with u_n increases in length by $2\delta/\sqrt{3}$ which implies that v moves downwards by 2δ.

We now apply a similar argument to T_v. Again, allow v to be a 'moveable' point, and consider the effect of changing the connection of the tree at one of the triples a_i, b_i, c_i (from the original connection as shown in Figure 5(a) to the alternative connection shown in Figure 5(b)) while keeping the tree a 3-direction Steiner tree. By the symmetry of the two connection types this is equivalent in its effect on v to moving x_i upwards by d_i in T_x; that is, v moves downwards by $2d_i$. This effect is additive across all of the triples, meaning that if we change to the alternative connection scheme at each $i \in J$ where $J \subseteq \{1, \ldots, n\}$ is a set solving the given instance of the SUBSET SUM PROBLEM, then v moves downwards by $2d$ to v_0, giving the required 3-direction minimum Steiner λ-tree on N_0.

Step 2: (2) ⇒ (1). The argument here is similar to that in Step 1. This time we begin with a 3-direction minimum Steiner λ-tree on N_0 with the same base topology as T_v, and treat the terminal v_0 as being a 'moveable' point on V_1. Since $v_0 \neq v$ it follows that there must be at least one $i \in \{1, \ldots, n\}$ such that the connection of the tree to a_i, b_i, c_i uses the alternative connection scheme shown in Figure 5(b). Let $J' \subseteq \{1, \ldots, n\}$ be the set of all such i where this alternative connection scheme is used. If for any $i \in J'$ we change to the original connection scheme (as shown in Figure 5(a)) while keeping the tree as a 3-direction tree then, by the same argument as in Step 1, v_0 moves upwards by $2d_i$. Now if for every $i \in J'$ we change to the original connection scheme while keeping the tree as a 3-direction tree then it is clear that v_0 now coincides with v (since the position of v_0 is uniquely determined by the positions of the other terminals, the topology of the tree and the three directions). Since $\mathbf{d}(v_0, v) = 2d$ it follows that $\sum_{i \in J'} d_i = d$, and hence J' gives a 'yes' solution to the given instance of the SUBSET SUM PROBLEM.

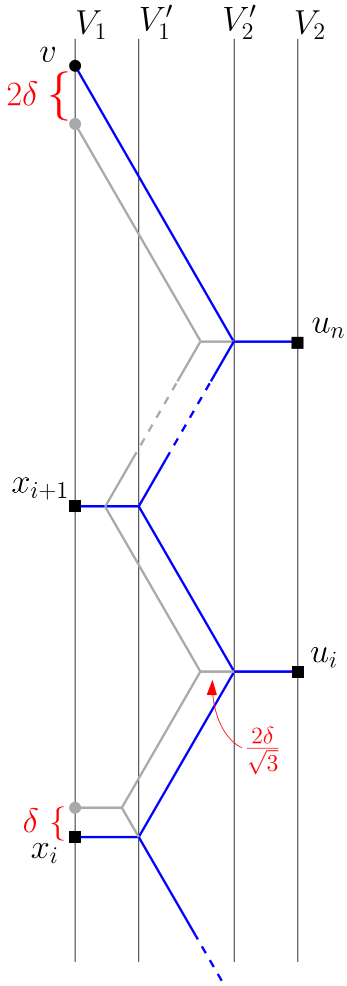

Figure 6: Construction for proof of Claim 3 (Step 1).

Step 3: (2) ⇒ (3). We first analyse the change in length to T_x under the movement of x_i by δ described in Step 1 and illustrated in Figure 6. For the horizontal edges: the edge incident with x_i decreases in length by $\delta/\sqrt{3}$; each edge incident with x_j for $i + 1 \leq j \leq n$ decreases in length by $2\delta/\sqrt{3}$; and each edge incident with u_j for $i \leq j \leq n$ increases in length by $2\delta/\sqrt{3}$. Hence the total length of the horizontal edges increases by $\delta/\sqrt{3}$. For the main zigzag path: its height decreases by 2δ and hence its length decreases by $4\delta/\sqrt{3}$. Together, these result in an overall decrease in length of $3\delta/\sqrt{3} = \sqrt{3}\delta$ for the whole tree.

It follows for the tree T_v that if we treat v as a 'moveable' point, and consider the effect of changing to the alternative connection of the tree at one of the triples a_i, b_i, c_i, while keeping the tree a 3-direction Steiner tree, the tree decreases in length by $\sqrt{3}d_i$. Hence, by additivity, the 3-direction minimum Steiner λ-tree on N_0 has length $L_v - \sqrt{3}d$.

Step 4: ¬(2) ⇒ ¬(3). To prove this last statement, we argue as follows: Suppose we have a Steiner tree T_0 on N_0 with the same base topology as T_v, but which is not a 3-direction tree. Colour all edges containing a horizontal component red. As described in Section 2.1, we can assume that there is only one bent edge; furthermore, the bent edge is the red edge connecting to the triple a_i, b_i, c_i. Now, suppose we replace this bent red edge by the orthogonal projection of this edge onto the line extending the horizontal component of the edge. It is easy to see, by the same argument as in Step 3, that the length of the resulting (disconnected) 3-direction λ-network is again $L_v - \sqrt{3}d$. The tree T_0 has length strictly longer than this, giving the required conclusion.

Discretisation and scaling. Above we have presented a transformation of any instance of the SUBSET SUM PROBLEM to show that the parallel lines λ-geometry Steiner tree problem is NP-hard if one ignores arithmetic precision issues. Furthermore, by the straightforward constructive nature of this transformation it is clear that the problem belongs to NP, and hence is NP-complete, up to issues of artihmetic precision. Here we demonstrate that the result remains true when applying a discretisation and scaling that resolves the issues related to computing with exact real arithmetic. A similar discretisation and scaling step has been described in detail in a number of previous papers (Brazil et al. 2000, Garey, Graham & Johnson 1977, Rubinstein et al. 1997), and so will only be sketched here.

In the discretisised problem Euclidean distances are rounded up to the nearest integer. Also, it is assumed that terminals and Steiner points can only have integer coordinates. Thus for a given Steiner tree T, performing discretisation increases or decreases the length of every edge by at most 3. Since all trees considered have at most $7n + 1$ edges, every tree is at most length $3 \cdot (7n + 1)$ longer or shorter than before the discretisation.

We need to be able to distinguish between 'yes' and 'no' instances in the discretisised problem. More precisely, as shown in the proof of Claim 3 above, we need to be able to distinguish between 3-direction minimum Steiner trees and non 3-direction minimum Steiner trees. The last part of the proof of Claim 3 shows that non 3-direction minimum Steiner trees have a length that is at least $\epsilon_\lambda = (1 - \cos\omega)/(2\sin\omega)$ above $L_v - \sqrt{3}d$, the length of 3-direction minimum Steiner trees.

The problem is now scaled by multiplying all terminal coordinates by an integer K. One can distinguish between 'yes' and 'no' instances, if $K\epsilon_\lambda - 2 \cdot 3 \cdot (7n + 1) \geq 1$. Choosing $K \geq (42n + 7)/\epsilon_\lambda$ suffices, and results in a polynomial scaling. ∎

Finally, since every instance of the parallel lines λ-geometry Steiner tree problem is also an instance of the λ-geometry Steiner tree problem, we immediately get the following corollary.

Corollary 3.2 *The λ-geometry Steiner tree problem is NP-complete for any given $\lambda = 3m$ (where m is a positive integer).*

4 Conclusion and Generalisations

The proof of Theorem 3.1 in the previous section relies, to a large extent, on the properties of the base tree T_x constructed in the course of the proof. A key property of the base tree is that if we perturb a single terminal x_i up or down along V_1 the resulting minimum Steiner tree on the new terminal set is strictly longer than T_x. If x_i is perturbed downwards (away from v) then the only change to the tree T_x is that the edge incident with x_i becomes a bent edge via the introduction of a new secondary direction; all other edges in the tree are straight primary edges. On the other hand, if x_i is perturbed upwards (towards v) then the edge incident with x_i again becomes a bent edge but this time via the introduction of a new primary direction; all other edges in the tree are straight secondary edges. This is possible due to the symmetry in the direction set for $\lambda = 3m$, which means that in a 3-direction Steiner tree such as T_x it is ambiguous as to whether the edges are all primary or all secondary (see Figure 2).

The difficulty in generalising Theorem 3.1 to other values of λ lies in the fact that the direction sets no longer exhibit this symmetry when $\lambda \neq 3m$. If we construct a base tree for one of these other values of λ (as in the proof of Theorem 3.1) using primary directions (as in the table in Figure 2) then it is no longer true that perturbing x_i in either direction along V_1 always reduces the length of the Steiner tree; for one of the two directions an edge other than the edge incident with x_i will become bent (i.e., the colour labeling changes), and it can be shown that the new minimum Steiner tree that results is shorter than the original base tree.

This problem, however, can be successfully circumvented via a slight alteration to the construction. If instead of choosing the lines V_1, V_1', V_2', V_2 to be vertical, we choose them to have a polar slope of $\pi/2 - \pi/(3\lambda)$, then it is possible to show that any perturbation of x_i along V_1 reduces the length of the corresponding base tree. The details of this somewhat technical argument will appear in a forthcoming paper (Brazil et al. 2012). The conclusion is that the parallel lines λ-geometry Steiner tree problem is NP-complete for all $\lambda > 2$.

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