Continuous-Trace C^*-Algebras and the Equivariant Brauer Group

Dana P. Williams

Dartmouth College

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Summary

- One the first goals in operator theory is to distinguish C^*-algebras from one another.
- For example, the classification program tries to classify large classes—for example, separable nuclear simple purely infinite C^*-algebras—by their K-theory.
- In these talks, I want to describe a very different sort of classification program for a class of C^*-algebras that are—in a sense to be described later—effectively locally trivial over their spectrum.
- One of the salient features is that we can parameterize this algebras via a group of invariants which allows us to use group theory in their study.
- After describing the classical theory, the goal will be to push the envelope to include group actions and crossed products.

Outline

- Today's talk will be mostly background.
- We'll start with C^*-bundles and $C_0(T)$-algebras.
- Then a review of Hilbert Modules and Morita equivalence.
- Then we'll recall the classical Dixmier-Douady classification of continuous-trace C^*-algebras and the classical Brauer group.
- Tomorrow we will get to the equivariant case by introducing the equivariant Brauer semigroup and the equivariant Brauer group.
- Then we will describe the structure of these objects and the subtle information they contain.
- Whenever they can be, our C^*-algebras will be separable and topological spaces Hausdorff and second countable.
- The structure of the equivariant Brauer group and related topics I'll discuss tomorrow is the work of many hands. Primary among them are David Crocker, Siegfried Echterhoff, Astrid an Huef, Alex Kumjian, Iain Raeburn, and myself.

The Basics

- So what is your favorite example of a C^*-algebra?
- If T is a locally compact Hausdorff space, then $C_0(T)$ is, up to isomorphism, the only example of a commutative C^*-algebra.
- If $\mathcal{H}$ is a complex Hilbert space, then the set, $B(\mathcal{H})$, of all bounded operators on $\mathcal{H}$ is a fundamental example.
- If $\mathcal{H}$ is finite dimensional, then we can identify $B(\mathcal{H})$ with the set $M_n(\mathbf{C})$ of complex $n \times n$-matrices.
- However, if $\mathcal{H}$ is infinite dimensional, $B(\mathcal{H})$ is not separable and is not amenable to elementary analysis.
- So instead, we usually consider the ideal $\mathcal{K}(\mathcal{H})$ of compact operators on $\mathcal{H}$ which is the C^*-algebra generated by the rank-one operators $\Theta_{h,k}$ where $\Theta_{h,k}(l) = (l \mid k)h$ for $h, k, l \in \mathcal{H}$.

Moving On

Example

Naturally, we can show off and combine these two basic examples into the C^*-algebra $A = C_0(T, \mathcal{K}(\mathcal{H}))$.

Remark

I would like to suggest using algebras like these to build more interesting examples, but we will want a bit more flexibility. A hint as to the next steps is that we can view $C_0(T, \mathcal{K}(\mathcal{H}))$ as the continuous sections $\Gamma_0(T \times \mathcal{K}(\mathcal{H}))$ where $T \times \mathcal{K}(\mathcal{H})$ is viewed as a trival bundle over T.

C^*-Bundles

Definition

A C^*-bundle over T is a continuous, open map $p : \mathscr{A} \to T$ where the fibres $A(t) = p^{-1}(t)$ are equipped with a C^*-algebra structure such that

1. $a \mapsto \|a\|$ is upper semicontinuous from $\mathscr{A}$ to $\mathbf{R}$,
2. $a \mapsto a^*$ is continuous from $\mathscr{A}$ to $\mathscr{A}$,
3. addition and multiplication are continuous from $\mathscr{A} \times_p \mathscr{A} \to \mathscr{A}$,
4. $(\lambda, a) \mapsto \lambda a$ is continuous from $\mathbf{C} \times \mathscr{A}$ to $\mathscr{A}$, and
5. if $\|a_i\| \to 0$ and $p(a_i) \to t \in T$, then $a_i \to 0_t$.

If $a \mapsto \|a\|$ is continuous, then we call $\mathscr{A}$ a continuous C^*-bundle over T. A continuous C^*-bundle $\mathscr{A}$ such that each fibre $A(t)$ is isomorphic to $\mathcal{K}(\mathcal{H}_t)$ for some Hilbert space $\mathcal{H}_t$ is called a elementary C^*-bundle.

Remark

If $p : \mathscr{A} \to T$ is a C^*-bundle, then $\Gamma_0(\mathscr{A})$ denotes the continuous sections of p vanishing at ∞; that is, continuous fucntions $f : T \to \mathscr{A}$ such that $p(f(t)) = t$. Then $\Gamma_0(\mathscr{A})$ is easily seen to be a $*$-algebra with respect to the obvious pointwise operations. Moreover it is complete with respect to the supremum norm $\|a\| := \sup_t \|a(t)\|$ and hence is a C^*-algebra.

Finding Bundles

Remark

While the C^*-bundle approach gives considerable insight, it is the section algebras that arise in practice.

Definition

A $C_0(T)$-algebra is a C^*-algebra A together with a non-degenerate homomorphism Φ from $C_0(T)$ into the center $ZM(A)$ of the multiplier algebra $M(A)$ of A.

Remark

We will usually suppress that map Φ and write $f \cdot a$ in place of $\Phi(f)a$. That is, we think of a $C_0(T)$-algebra A as a $C_0(T)$-bimodule where $f \cdot a = a \cdot f$ for $a \in A$ and $f \in C_0(T)$.

Back to Bundles

Remark

Suppose that A is a $C_0(T)$-algebra. If $t \in T$, let J_t be the maximal ideal of $C_0(T)$ of functions vanishing at t, and let I_t be the ideal of A generated by $J_t \cdot A$. If $a \in A$, let $a(t)$ be the image of a in the quotient $A(t) = A/I_t$. The map $t \mapsto \|a(t)\|$ is upper semicontinuous. Then we can view each $a \in A$ as giving a section of $\mathscr{A} = \coprod A(t)$ as a bundle over T. If $P \in \operatorname{Prim} A$, then there is a unique t such that $P \supset I_t$ giving a continuous map $\sigma_A : \operatorname{Prim} A \to T$.

The Result

Theorem

The following are equivalent

1. A *is a* $C_0(T)$*-algebra.*
2. *There is a continuous map* $\sigma_A : \operatorname{Prim} A \to T$*.*
3. *There is a* C^**-bundle* $\mathscr{A}$ *over* T *and a* $C_0(T)$*-linear isomorphism of* A *onto* $\Gamma_0(\mathscr{A})$*.*

The map σ_A *is open if and only if* $\mathscr{A}$ *is a continuous* C^**-bundle.*

Idea of the Proof.

If $\mathscr{A}$ is a C^*-bundle over T, then $\Gamma_0(\mathscr{A})$ is a $C_0(T)$-algebra. Given a $C_0(T)$-algebra, we described the map $\sigma_A : \operatorname{Prim} A \to T$ on the previous slide. Given a map σ_A as above, the Dauns-Hoffman Theorem gives an map of $C_b(\operatorname{Prim} A)$ into the center of $M(A)$ (for any C^*-algebra) and then we get a map on $C_0(T)$ via composition with σ_A. If A is a $C_0(T)$ algebra, then our C^*-bundle is formed from $\coprod A(t)$. $\square$

"Trivial" Examples

Example

For the moment, we want to keep things very elementary. Let $\mathscr{U} = \{ U_1, \dots, U_n \}$ be an open cover of T, and let $U_{ij} := U_i \cap U_j$. Define

$$A(\mathscr{U}) := \{ a \in M_n(C_0(T)) : \operatorname{supp} a_{ij} \subset U_{ij} \}.$$

Then $A(\mathscr{U})$ is the section algebra of an elementary C^*-bundle over T with fibres $A(\mathscr{U})(t) = M_{n_t}$ where $n_t = \#\{ i : t \in U_i \}$. More generally, we could let $\mathscr{U}$ be a locally finite cover. Return

Example

Let $T = [0,1]$ and $\mathscr{U} = \{ [0,1], [0,1) \}$. Then

$$A(\mathscr{U}) = \{ f \in C(T, M_2(\mathbf{C})) : f(1) = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \}$$

Why Trivial?

Remark

I want to think of $A(\mathscr{U})$ on the previous slide as a trivial object as they are the "just compact operators" on a generalized Hilbert space. Informally, let

$$\mathsf{X} := \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in C_0(T) \text{ and } \operatorname{supp} x_i \subset U_i \right\}$$

Then $C_0(T)$ acts on the right of X and we can equip X with a $C_0(T)$-valued inner product $\langle x \, , \, y \rangle_{C_0(T)}(t) := x(t)^* y(t) = \sum_i \overline{x_i(t)} y_i(t)$. The corresponding "rank-one" operators on X are given by $\Theta_{x,y}(z) := x \cdot \langle y \, , \, z \rangle_{C_0(T)} = xy^* z$. Then we can identify $\Theta_{x,y}$ with matrix multiplication by the element of $A(\mathscr{U})$ given by ${}_{A(\mathscr{U})}\langle x \, , \, y \rangle = xy^*$; that is ${}_{A(\mathscr{U})}\langle x \, , \, y \rangle(t)$ is the element of $M_n\big(C_0(T)\big)$ with $(i,j)^{\mathrm{th}}$ entry $x_i(t)\overline{y_j(t)}$.

Thus, we think of $A(\mathscr{U})$ as an algebra of "generalized compact operators" on X.

Definition

Let B be a C^*-algebra. A right Hilbert B-module is a right B-module X equipped with a B-valued sequilinear form such that

1. $\langle x\,,\,y\cdot b\rangle_B = \langle x\,,\,y\rangle_B b$,
2. $\langle x\,,\,y\rangle_B^* = \langle y\,,\,x\rangle_B$,
3. $\langle x\,,\,x\rangle_B$ is positive in B, and
4. X is complete with respect to the norm $\|x\| := \|\langle x\,,\,x\rangle_B\|^{\frac{1}{2}}$.

We call X full if the span of $\langle \cdot\,,\,\cdot\rangle_B$ is dense in B.

Remark

We can equally well work with left Hilbert B-modules. In that case, X is a left B-module and we replace (1) above with

1. ${}_B\langle b\cdot x\,,\,y\rangle = b\,{}_B\langle x\,,\,y\rangle$.

Operators on X

Just as with Hilbert spaces, it is the operators on X that are interesting.

Definition

A linear operator $T : \mathsf{X} \to \mathsf{X}$ on a right Hilbert module X is called adjointable of there is a $T^* : \mathsf{X} \to \mathsf{X}$ such that $\langle Tx\ , \ y\rangle_{B} = \langle x\ , \ T^*y\rangle_{B}$ for all $x, y \in X$. The collection $L(\mathsf{X})$ of adjointable operators is a C^*-algebra with respect to the operator norm.

Definition

The generalized compact operators on a (right) Hilbert module X is the ideal $K(\mathsf{X})$ in $L(\mathsf{X})$ formed by the closed span of the rank-one operators $\Theta_{x,y}$ with $x, y \in \mathsf{X}$: $\Theta_{x,y}(z) := x \cdot \langle y\ , \ z\rangle_{B}$.

Morita Equivalence

Definition

Two C^*-algebras A and B are Morita equivalent if there is a a $A-B$-bimodule X, called an imprimitivity bimodule, such that

1. X is both a full left Hilbert A-module and a full right Hilbert B-module,
2. $\langle a \cdot x \,, y\rangle_B = \langle x \,, a^* \cdot y\rangle_B$ and ${}_A\langle x \,, y \cdot b\rangle = {}_A\langle x \cdot b^* \,, y\rangle$ for all $x, y \in \mathsf{X}$, $a \in A$, and $b \in B$, and
3. $x \cdot \langle y \,, z\rangle_B = {}_A\langle x \,, y\rangle \cdot z$ for $x, y, z \in \mathsf{X}$.

Theorem

A is Morita equivalent to B if and only if there is a right Hilbert B-module X such that A is isomorphic to $K(\mathsf{X})$. (In this case, X becomes a left Hilbert A-module with respect to ${}_A\langle x \,, y\rangle := \varphi^{-1}(\Theta_{x,y})$.)

Theorem (Brown-Green-Rieffel)

Two separable C^*-algebras A and B are Morita equivalent if and only if $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ are isomorphic, where here and elsewhere, $\mathcal{K}$ denotes the algebra of compact operators on a separable infinite-dimensional Hilbert space.

Remark

Suppose that $\mathscr{A}$ is an elementary C^*-bundle over T and that $A = \Gamma_0(\mathscr{A})$. Then $A \otimes \mathcal{K}$ is the section algebra $\Gamma_0(\mathscr{A} \otimes \mathcal{K})$ of an elementary C^*-bundle $\mathscr{A} \otimes \mathcal{K}$ over T where we have "simply" tensored each elementary fibre, $A(t)$ with $\mathcal{K}$. Now every fibre is (isomorphic to) $\mathcal{K}$. But it usually will not be the case that $A \otimes \mathcal{K} \cong C_0(T, \mathcal{K})$—even locally.

Local Triviality

Proposition

Suppose that $A = \Gamma_0(\mathscr{A})$ *for an elementary* C^**-bundle over* T*. Then* $A \otimes \mathcal{K} \cong C_0(T, \mathcal{K})$ *if and only if* A *is Morita equivalent to* $C_0(T)$*.*

Definition

A C^*-algebra with (Hausdorff) spectrum T has continuous trace if every point $t \in T$ has an open neighborhood U such that the corresponding ideal $A(U)$ is Morita equivalent to $C_0(U)$. That is, A has continuous trace if it has Hausdorff spectrum and is locally Morita equivalent to a commutative C^*-algebra.

Remark

In bundle terms, A has continuous trace if $A \otimes \mathcal{K}$ is locally trivial. Alternatively, $\mathscr{A} \otimes \mathcal{K}$ is locally trivial.

A Technicality

- Suppose that $B = \Gamma_0(\mathscr{B})$ is given by an elementary C^*-bundle $\mathscr{B}$ over T.
- If X is a right Hilbert B-module, then it turns out that $K(\mathsf{X}) = \Gamma_0(\mathscr{C})$ for an elementary C^*-bundle $\mathscr{C}$ over T.
- Thus if $A = \Gamma_0(\mathscr{A})$ is Morita equivalent to B, then A inherits a $C_0(T)$-module structure from its isomorphism with $K(\mathsf{X})$.
- But this may not be the same as its natural module structure as a section algebra.
- When the isomorphism of A with $\mathcal{K}(\mathsf{X})$ can be taken so that the module structures coincide, then we say that A and B are Morita equivalent over T.
- We let $\mathrm{Br}(T)$ be the collection of Morita equivalence classes over T. return

$H^3(T)$

- We want to show that $\mathrm{Br}(T)$ can be identified with the third cohomology group $H^3(T)$ of the space T.
- Here it will suffice to know that $H^3(T)$ is (isomorphic to) the sheaf group $H^2(T, \mathcal{S})$.

If $\mathscr{U} = \{ U_i \}$ is an open cover of T, then a 2-cocycle $\nu \in Z^2(\mathscr{U}, \mathcal{S})$ is a collection $\nu = \{ \nu_{ijk} \}$ of functions $\nu_{ijk} : U_{ijk} \to \mathbf{T}$ on triple overlaps such that

$$\nu_{ijk}(t)\nu_{ikl}(t) = \nu_{ijl}(t)\nu_{jkl}(t) \quad \text{for } t \in U_{ijkl}.$$

We say that $\nu \in B^2(\mathscr{U}, \mathcal{S})$ if there are functions $\mu_{ij} : U_{ij} \to \mathbf{T}$ such that $\nu_{ijk}(t) = \mu_{ij}(t)\mu_{jk}(t)\overline{\mu_{ik}(t)}$. Note that both $Z^2(\mathscr{U}, \mathcal{S})$ and $B^2(\mathscr{U}, \mathcal{S})$ are abelian groups (pointwise multiplication) so we can form the quotient $H^2(\mathscr{U}, \mathcal{S})$. Then $H^2(T, \mathcal{S})$ is the direct limit of the groups $H^2(\mathscr{U}, \mathcal{S})$ where the covers $\mathscr{U}$ are directed by refinement.

The Raeburn-Taylor Algebras

- Given $\nu = \{ \nu_{ijk} \} \in Z^2(\mathscr{U}, \mathcal{S})$, we can make a more interesting C^*-algebra out of our $A(\mathscr{U})$'s defined earlier.
- The 2-cocycle property allows us to redefine or twist the multiplication in $A(\mathscr{U})$ as follows: if $(a_{ij}), (b_{ij}) \in A(\mathscr{U})$, then we define $(a_{ij})(b_{ij}) = (c_{ij})$ where

$$c_{ij}(t) = \begin{cases} \sum_{\{ l : t \in U_{ilj} \}} \overline{\nu_{ilj}(t)} a_{il}(t) b_{lj}(t) & \text{if } t \in U_{ij}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

- We can define an involution by $(a_{ij})^* = (\overline{a_{ji}})$.
- Then the resulting so-called "Raeburn-Taylor" C^*-algebra is denoted by $A(\mathscr{U}, \nu)$.

Remark (The Fine Print)

In order for the above $*$-operation to truly be an involution, we must work with an *alternating* cocycle ν: we need $\nu_{\sigma(i)\sigma(j)\sigma(k)} = (\nu_{ijk})^{\mathrm{sgn}(\sigma)}$ for all permutations σ of $\{ i, j, k \}$. Fortunately, every class in $H^2(\mathscr{U}, \mathcal{S})$ has such a representative, so this is not worth worrying about much here.

Remark

If $\nu \in B^2(\mathscr{U}, \mathcal{S})$, then $A(\mathscr{U}, \nu) \cong A(\mathscr{U})$. To see this, suppose that $\nu = \partial\mu$ (so that $\nu_{ijk} = \mu_{ij}\mu_{jk}\overline{\mu_{ik}}$), then $(a_{ij}) \mapsto (\mu_{ij}a_{ij})$ does the job.

Remark

If $\nu \in Z^2(\mathscr{U}, \mathcal{S})$ and if $U_l \in \mathscr{U}$, then $\nu|_{U_l} = \{\, \nu_{ijk}|_{U_l} \,\}$ is in $B^2(\mathscr{U}|_{U_l}, \mathcal{S})$. (Let $\mu_{ij} = \nu_{lij}$. Then $\nu_{ijk} = \partial\mu_{ijk}$ for any $x \in U_{ijk} \cap U_l$.) It follows that $A(\mathscr{U}, \nu)|_{U_l} \cong A(\mathscr{U}|_{U_l})$ is Morita equivalent to $C_0(U_l)$. Therefore, $A(\mathscr{U}, \nu)$ has continuous-trace with spectrum T.

The Dixmier-Douady Class

Theorem (Dixmier-Douady)

Every continuous trace C^*-algebra A with spectrum T is Morita equivalent (over T) to an $A(\mathscr{U}, \nu)$ for some $\nu \in Z^2(\mathscr{U}, \mathcal{S})$. The class of ν in $H^2(T, \mathcal{S})$ depends only on the Morita equivalence class of A and is called the Dixmier-Douady class, $\delta(A)$, of A. Two continuous-trace C^*-algebras with spectrum T are Morita equivalent (over T) if and only if $\delta(A) = \delta(B)$. Therefore $A \mapsto \delta(A)$ induces a bijection of $\mathrm{Br}(T)$ onto $H^3(T)$.

Oversimplified Proof.

The idea of the proof is that the obstruction to "gluing" together the local Morita equivalences of A with $C_0(U_i)$ to build a global Morita equivalence with $C_0(T)$ is naturally given by a 2-cocycle $\nu = \{\, \nu_{ijk} \,\} \in Z(\mathscr{U}, \mathcal{S})$. □

Break

Let's Take a Short Break

The Brauer Semigroup

- With a view to generalizations, let's pause to introduce some objects we will need down the road.
- Let $\mathscr{S}(T)$ be the collection of $C_0(T)$-algebras.
- We want to use a balanced tensor product on $\mathscr{S}(T)$ to form a semigroup with identity $C_0(T)$.
- Since $\mathscr{S}(T)$ certainly contains non-nuclear C^*-algebras, some care will be needed.

Definition

If $A, B \in \mathscr{S}(T)$, then the balanced tensor product $A \otimes_{C_0(T)} B$ is the quotient of $A \otimes_{\max} B$ by the ideal generated by elements of the form $a \cdot f \otimes b - a \otimes f \cdot b$ with $a \in A$, $b \in B$, and $f \in C_0(T)$.

Remark

If we try the same construction with $\otimes_{\min}$, Blanchard has observed that we do not necessarily obtain an associative operation.

Example

If $A = \Gamma_0(\mathscr{A})$ and $B = \Gamma_0(\mathscr{B})$ are continuous-trace C^*-algebras with spectrum T, then $A \otimes_{C_0(T)} B$ is the continuous-trace C^*-algebra $\Gamma_0(\mathscr{A} \otimes_\Delta \mathscr{B})$ where $\mathscr{A} \otimes_\Delta \mathscr{B}$ is the restriction of the bundle $\mathscr{A} \times \mathscr{B}$ to the diagonal $\Delta = \{\, (t,t) : t \in T \,\}$.

Group Structure

Proposition

If A and B are continuous-trace C^*-algebras with spectrum T, then

$$\delta(A \otimes_{C_0(T)} B) = \delta(A) + \delta(B). \qquad (\ddagger)$$

In particular, $\mathrm{Br}(T)$ *is a group with respect to* $\otimes_{C_0(T)}$ *and* $\mathrm{Br}(T) \cong H^3(T)$. *(Here we are viewing $\delta(A)$ as an element in the additive group $H^3(T)$.)*

Sketch of Proof.

We just need to prove $(\ddagger)$. We'll consider only the case $A = A(\mathscr{U}, \nu)$ and $B = A(\mathscr{V}, \rho)$. Let $\mathscr{U} \otimes \mathscr{V} = \{ U_i \cup V_j \}_{i=1,j=1}^{n,m}$, and let ν^* and ρ^* be the images of ν and ρ in $Z^2(\mathscr{U} \otimes \mathscr{V}, \mathcal{S})$. Then we can define an isomorphism of $A(\mathscr{U}, \nu) \otimes_{C_0(T)} A(\mathscr{V}, \rho)$ to $A(\mathscr{U} \otimes \mathscr{V}, \nu^* \rho^*)$ by $(a_{ij}) \otimes (b_{kl}) \mapsto (c_{ik,jl})$, where $c_{ik,jl} = a_{ij} b_{kl}$. The details are messy, but straightforward. $\square$

More on $\mathrm{Br}(T)$

Remark

The identity element of $\mathrm{Br}(T)$ is just the class of $C_0(T)$, or equivalently, $C_0(T,\mathcal{K})$. The inverse of $[A] \in \mathrm{Br}(T)$ is given by $[\overline{A}]$, where $\overline{A}$ is the *conjugate algebra* $\overline{A} = \{\, a^\flat : a \in A \,\}$ which has the same operations as does A except for scalar multiplication: $\lambda a^\flat := (\bar{\lambda} a)^\flat$. return

Dynamical Systems

- Now it is time to generalize the classical Brauer group to include a group action.
- Recall that a C^*-dynamical system (A, G, α) consists of a C^*-algebra A, a locally compact group G, and a homomorphism $\alpha : G \to \operatorname{Aut} A$ such that $s \mapsto \alpha_s(a)$ is continuous for all $a \in A$.
- A covariant homomorphism of (A, G, α) into the multiplier algebra $M(B)$ of a C^*-algebra B is a pair (π, u) consisting of a nondegenerate homomorphism $\pi : A \to M(B)$ and a strictly continuous unitary-valued homomorphism $u : G \to M(B)$ such that $\pi\big(\alpha_s(a)\big) = u_s\pi(a)u_s^*$.
- If $B = B(\mathcal{H}) = M(\mathcal{K}(\mathcal{H}))$, then we call (π, u) a covariant representation of (A, G, α).

Crossed Products

- Given a dynamical system (A, G, α), we can form the crossed product $A \rtimes G$ which should be viewed as a sort of universal object for covariant representations.
- The crossed product is characterized by the following:
  1. There is a covariant homomorphism (i_A, i_G) of (A, G, α) into $M(A \rtimes_\alpha G)$.
  2. If (π, u) is a covariant homomorphism, there is a unique homomorphism $\pi \rtimes u$ such that $(\pi \rtimes u) \circ i_A = \pi$ and $(\pi \rtimes u) \circ i_G = u$.
  3. $A \rtimes_\alpha G = \overline{\operatorname{span}}\{\, i_A(a) i_G(z) : \text{for } a \in A \text{ and } z \in C_c(G)\,\}$.

Remark

Of course, the theory would be useless without knowing that there exist convariant representations. But if $\pi : A \to B(\mathcal{H})$ is a representation, then we get a covariant representation $(\overline{\pi}, \lambda)$ on $L^2(G, \mathcal{H})$ where $\overline{\pi}(a)h(s) = \pi\bigl(\alpha_s^{-1}(a)\bigr)\bigl(h(s)\bigr)$ and $\lambda_r h(s) = h(r^{-1}s)$. If π and ρ are both faithful representations of A, then $\ker\bigl(\overline{\pi} \rtimes \lambda\bigr) = \ker\bigl(\overline{\rho} \rtimes \lambda\bigr)$. The quotient of $A \rtimes_\alpha G$ by this common kernel is the reduced crossed product $A \rtimes_{\alpha,r} G$.

Morita Equivalent Systems

Definition

We say that two dynamical systems (A, G, α) and (B, G, β) are Morita equivalent if A and B are Morita equivalent via a bimodule X and there is a strongly continuous action u of G on X such that

$$ {}_A\big\langle u_s(x)\,,\, u_s(y)\big\rangle = \alpha_s\big({}_A\langle x\,,\, y\rangle\big) \quad \text{and} $$
$$ \big\langle u_s(x)\,,\, u_s(y)\big\rangle_B = \beta_s\big(\langle x\,,\, y\rangle_B\big). $$

Examples

- Recall that two G-actions α and β on A are exterior equivalent if there is a strictly continuous unitary-valued map $v : G \to M(A)$ such that $\beta_s(a) = v_s \alpha_s(a) v_s^*$ and $v_{st} = v_s \overline{\alpha}_s(v_t)$.
- Two actions α and β on different C^*-algebras are cocycle conjugate if there is an isomorphism $\Phi : A \to B$ such that α and $\Phi \circ \beta \circ \Phi^{-1}$ are exterior equivalent.
- Combes showed that that α and β as above are Morita equivalent if and only if $\alpha \otimes 1$ on $A \otimes \mathcal{K}$ and $\beta \otimes 1$ on $B \otimes \mathcal{K}$ are exterior equivalent.

As Expected

Theorem

If (X, u) *implements a Morita equivalence between* (A, G, α) *and* (B, G, β)*, then* $C_c(G, \mathsf{X})$ *admits a completion* $\mathsf{X} \rtimes_u G$ *that implements a Morita equivalence between* $A \rtimes_\alpha G$ *and* $B \rtimes_\beta G$*.*

Definition

Suppose that (A, G, α) is a dynamical system, that A is a $C_0(T)$-algebra, and that T is a G-space. Then we say that α covers the given action on T if

$$\alpha_s(f \cdot a) = \mathrm{lt}_s(f) \cdot \alpha_s(a) \quad \text{for all } s \in G,\ f \in C_0(T), \text{ and } a \in A$$

where $\mathrm{lt}_s(f)(t) = f(s^{-1} \cdot t)$.

Definition

We assume that T is a G-space. We let $\mathscr{S}_G(T)$ be the collection of all pairs (A, α) such that α covers the given action. Kasparov called the elements of $\mathscr{S}_G(T)$ G–$C_0(T)$-algebras.

What's in $\mathscr{S}_G(T)$?

Remark

Notice that $\mathscr{S}_G(T)$ is never empty. For example, $\bigl(C_0(T), \mathrm{lt}\bigr) \in \mathscr{S}_G(T)$. More generally, if (D, G, γ) is any dynamical system and we give $A = C_0(T, D)$ the obvious $C_0(T)$-algebra structure, then $\bigl(C_0(T, D), \mathrm{lt} \otimes \gamma\bigr) \in \mathscr{S}_G(T)$. However, even if A itself has spectrum T, it is a very difficult problem as to whether or not A admits an action α covering the given G-action on T. We will have more to say about this tomorrow.

- Suppose that (A, α) and (B, β) in $\mathscr{S}_G(T)$ are Morita equivalent via (X, u).
- The the A and B actions on X extend to the multiplier algebras and X becomes a $C_0(T)$-bimodule.
- The Rieffel homeomorphism $h_{\mathsf{X}} : \operatorname{Prim} A \to \operatorname{Prim} B$ then

$$\begin{tikzcd} \operatorname{Prim} A \arrow[rr, "h_{\mathsf{X}}"] \arrow[dr, "\sigma_A"'] & & \operatorname{Prim} B \arrow[dl, "\sigma_B"] \\ & T & \end{tikzcd}$$

  commutes if and only if $f \cdot x = x \cdot f$ for all $x \in \mathsf{X}$ and $f \in C_0(T)$.
- In this case, we say that (A, α) and (B, β) are Morita equivalent over T.
- If $G = \{ e \}$ and A and B have continuous trace with specturm T, then this reduces to the equivalence relation in $\operatorname{Br}(T)$. go.

Definition

We let $S_G(T)$ be the set of Morita equivalence classes over T in $\mathscr{S}_G(T)$. We call $S_G(T)$ the equivariant Brauer semigroup. The class of $(A, \alpha) \in \mathscr{S}_G(T)$ in $S_G(T)$ is denoted by $[A, \alpha]$.

Remark

If (A, α) and (B, β) are equivalent in $\mathscr{S}_G(T)$ and if A has continuous trace with spectrum T, then so does B. Hence we can make the following definition.

Definition

We let $\mathrm{Br}_G(T)$ be the subset of $S_G(T)$ consisting of classes $[A, \alpha]$ with A having continuous-trace with spectrum T. We call $\mathrm{Br}_G(T)$ the equivariant Brauer group.

The Product

Remark

Suppose that (A, α) and (B, β) are in $\mathscr{S}_G(T)$. Then $\alpha \otimes \beta$ gives an action on $A \otimes_{\max} B$ that factors through an action $\alpha \otimes_{C_0(T)} \beta$ $A \otimes_{C_0(T)} B$. It is not hard to see that $\alpha \otimes_{C_0(T)} \beta$ also covers the given G-action on T.

Proposition

The binary operation

$$[A, \alpha] \cdot [B, \beta] := [A \otimes_{C_0(T)} B, \alpha \otimes_{C_0(T)} \beta]$$

is well-defined on $S_G(T)$, and so equipped, $S_G(T)$ is a commutative semigroup with identity equal to the class of $(C_0(T), \mathrm{lt})$. We call $S_G(T)$ the equivariant Brauer semigroup.

Moving Forward

Remark

We will only be able to say something significant about $S_G(T)$ under strong hypotheses for the given action of G on T. Since $S_G(T)$ is a semigroup with an identity, the group of invertibles in $S_G(T)$ is a much more tractible object—since the group structure makes it much more amenable to calculation.

Until Tomorrow

- That is enough for one day.
- Now that we have the preliminaries in hand, tomorrow we will examine the structure of $\mathrm{Br}_G(T)$ and $S_G(T)$ as best we can.
- As we will see, $\mathrm{Br}_G(T)$ especially has a tantalizing rich structure.

References

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The Equivariant Brauer Group

Theorem

The group of invertible elements in $S_G(T)$ is exactly the equivariant Brauer group $\mathrm{Br}_G(T)$.

Proof.

If $[A, \alpha]$ is invertible in $S_G(T)$, then there is a $B \in \mathscr{S}(T)$ such that $A \otimes_{C_0(T)} B$ is Morita equivalent to $C_0(T)$. A nontrivial result going back to Green implies that in this case both A and B have continuous trace with spectrum T. This means that $[A, \alpha] \in \mathrm{Br}_G(T)$.

The Converse

Proof Continued.

Now suppose that $[A, \alpha] \in \mathrm{Br}_G(T)$. We have to exhibit and inverse. As we saw earlier go, the conjugate algebra $\overline{A} = \{ a^\flat : a \in A \}$ is an inverse for A in $\mathrm{Br}(T)$. Thus there is a $A\text{–}C_0(T)$-bimodule X implementing a Morita equivalence between A and $C_0(T)$. We get an element $(\overline{A}, \overline{\alpha}) \in \mathscr{S}_G(T)$ via $\overline{\alpha}_s(a^\flat) = \alpha_s(a)^\flat$. We claim that $[\overline{A}, \overline{\alpha}]$ is our desired inverse. This is not so easy. We get that $\mathsf{X} \otimes_{C_0(T)} \overline{\mathsf{X}}$ is a $A \otimes_{C_0(T)} \overline{A}\text{–}C_0(T)$ bimodule and that it implements a Morita equivalence.

Proof

Proof Continued.

Unfortunately, $\mathsf{X} \otimes_{C_0(T)} \overline{\mathsf{X}}$ carries no obvious G-action—let alone one covering left-translation on $C_0(T)$. But one can show that $\mathsf{X} \otimes_{C_0(T)} \overline{\mathsf{X}}$ is isomorphic (as a bimodule) to (the completion of)

$$\mathsf{N} = \{\, a \in A : t \mapsto \operatorname{tr}(a^* a(t)) \in C_0(T) \,\}.$$

(Recall that the fibres of A are elementary C^*-algebras and there is a well-defined trace of the positive elements of each fibre.) Unlike $\mathsf{X} \otimes_{C_0(T)} \overline{\mathsf{X}}$, N has a natural G-action: $u_s(a) = \alpha_s(a)$. Now it is possible to show that (N, u) implements a Morita equivalence between $(A \otimes_{C_0(T)} \overline{A}, \beta)$ and $(C_0(T), \mathrm{lt})$. Some serious untangling shows $\beta = \alpha \otimes_{C_0(T)} \overline{\alpha}$. □